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NORMAL NUMBERS AND SELECTION RULES

BY

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ABSTRACT

Given a *normal number* $x = 0, x_1x_2\cdots$ to base 2 and a *selection rule* $S \subset \{0,1\}^* = \cup_{n=0}^{\infty}\{0,1\}^n$, we define a subsequence $x_s = 0, x_{t_1}x_{t_2}\cdots$ where $\{t_1 < t_2 < \cdots\} = \{i;\, x_1x_2\cdots x_{i-1} \in S\}$. x_s is called a *proper* subsequence of x if $\overline{\lim}_{i\to\infty} t_i/i < \infty$. A selection rule S is said to *preserve normality* if for any normal number x such that x_s is a proper subsequence of x, x_s is also a normal number. We prove that if $S/\sim_s$ is a finite set, where $\sim_s$ is an equivalence relation on $\{0,1\}^*$ such that $\xi \sim_s \eta$ if and only if $\{\zeta;\, \xi\zeta \in S\} = \{\zeta;\, \eta\zeta \in S\}$, then S preserves normality. This is a generalization of the known result in finite automata case, where $\{0,1\}^*/\sim_s$ is a finite set (Agafonov [1]).

1. Introduction

Let $x = 0,\ x_1x_2\cdots$ be the binary expansion of a real number $x \in [0,1)$. x is called a *normal number* (to base 2) if for any $k \in N = \{1,2,\cdots\}$ and $\xi = (\xi_1, \xi_2, \cdots, \xi_k) \in \{0,1\}^k$,

$$\lim_{n\to\infty}\frac{1}{n}\left|\{i;\, x_i = \xi_1, x_{i+1} = \xi_2, \cdots, x_{i+k-1} = \xi_k,\ 1 \leqq i \leqq n\}\right| = 2^{-k}.$$

Let $\{0,1\}^* = \bigcup_{k=0}^{\infty}\{0,1\}^k$, where $\{0,1\}^0 = \{\wedge\}$ and $\wedge$ is the *empty word*. For $\xi = (\xi_1, \xi_2, \cdots, \xi_k)$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_l)$ both belonging to $\{0,1\}^*$, define the *concatenation* $\xi\eta \in \{0,1\}^*$ by $\xi\eta = (\xi_1, \xi_2, \cdots, \xi_k, \eta_1, \eta_2, \cdots, \eta_l)$. Thus $\{0,1\}^*$ is considered as a semigroup under concatenation. For $S \subset \{0,1\}^*$, define an equivalence relation $\sim_s$ on $\{0,1\}^*$ as follows: $\xi \sim_s \eta$ if $\{\zeta;\, \xi\zeta \in S\} = \{\zeta;\, \eta\zeta \in S\}$. Note that S is a union of equivalence classes and that if $\xi \sim_s \eta$, then $\xi\zeta \sim_s \eta\zeta$ for any $\zeta \in \{0,1\}^*$.

A subset S of $\{0,1\}^*$ is called a *selection rule*. For a selection rule S and a real number $x = 0, x_1x_2\cdots$ (2-expansion), we define $x_s = 0, x_{t_1}x_{t_2}\cdots$, where $\{t_1 < t_2 < \cdots\} = \{i;\, (x_1, x_2, \cdots, x_{i-1}) \in S\}$. We call x_s a *proper* subsequence of x if $\overline{\lim}_{i\to\infty} t_i/i < \infty$. A selection rule S is said to *preserve normality* if for any

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normal number x such that x_s is a proper subsequence of x, x_s is also a normal number.

We are interested in determining which selection rules preserve normality and which selection rules do not preserve normality. We have already shown that selection rules of the form $S = \bigcup_{i=1}^{\infty} \{0, 1\}^{n_i}$ preserve normality if and only if the sequence $n_1 < n_2 < \cdots$ is *completely deterministic* ([2], [3]).

In this paper, we prove that if $S/\sim_s$ is a finite set, then S preserves normality. This result contains the following two special cases:

1. The finite automata case, where $\{0, 1\}^*/\sim_s$ is a finite set. The result here was obtained by Agafonov [1].

2. The renewal case, where $S/\sim_s$ consists of one element. A similar result of this special case under some additional restriction was stated in [3].

We will also give some examples of selection rules which preserve normality, and some examples of rules that don't. Of course the results obtained here carry over immediately to any finite state Bernoulli shift. The restriction to the 2-shift is for convenience only.

We will use another equivalent definition of normal number. Let P be the uniform distribution on $\{0, 1\}$; $P(0) = P(1) = \frac{1}{2}$. A sequence $\alpha \in \{0, 1\}^N$ is called a *normal number* if α is *generic* for the product measure P^N with respect to the shift; $\mu_\alpha = P^N$ in the notation used in [2]. The notation and terminologies used here follow those in [2].

2. Proof of main theorem

THEOREM. *If $S/\sim_s$ is a finite set, then the selection rule S preserves normality.*

PROOF. Let S be a selection rule such that $S/\sim_s$ is a finite set. Let $\alpha \in \{0, 1\}^N$ be any normal number such that α_s is a proper subsequence of α. Let

$$W = \{i\,;\,(\alpha(1), \alpha(2), \cdots, \alpha(i-1)) \in S\}$$

$$= \{t_1 < t_2 < \cdots\}.$$

Then $\underline{\sigma}(W) > 0$ from the assumption that α_s is a proper subsequence. To prove that α_s is a normal number, it is sufficient to prove that for any finite subset V' of N, there exists $V \subset V'$ such that $\mu_{\alpha_s}^V = P^N$.

We first prove this fact under an additional hypothesis on V', and then show how to reduce the general case to the case considered here.

LEMMA 1. *Let S be the selection rule such that $S/\sim_s$ is a finite set. Let α be a normal number such that α_s is a proper subsequence. Let*

$$W = \{i\,;\,(\alpha(1), \alpha(2), \cdots, \alpha(i-1)) \in S\}$$
$$= \{t_1 < t_2 < \cdots\}.$$

Then for any infinite subset V of N such that

$$\lim_{k\to\infty} \sigma U'\left(\bigcup_{i=0}^{k} (W - i)\right) = 1,$$

where $U' = \{t_i\,;\, i \in V'\}$ and $W - i = \{j\,;\, j + i \in W\}$, there exists $V \subset V'$ such that $\mu_{\alpha_s}^{V} = P^N$.

PROOF. Let $\Sigma = \{0,1\}^*/\sim_s$ and $K = S/\sim_s$. Σ is considered as a topological space with the discrete topology. Let F be a subset of $\{0,1\}^N \times \Sigma^N$ such that

$$F = \{(\gamma, \tau);\ \tau(i+1) = \tau(i)\gamma(i) \text{ for any } i \in N\}.$$

Then F is a closed shift invariant set. Define $\beta \in \Sigma^N$ by $\alpha(1)\alpha(2)\cdots\alpha(i-1) = \beta(i)$. We can find a subset U of U' such that

$$\mu = \text{vague}\ \underset{\substack{n\in U\\ n\to\infty}}{\text{limit}}\ \frac{1}{n}\sum_{i=0}^{n-1} \delta_{(T^i\alpha, T^i\beta)}$$

exist, since $(\alpha, \beta) \in F$ and F is a countable union of compact sets of the form $\{(\gamma, \tau);\ \tau(1)$ belongs to a finite set of $\Sigma\}$. Let $K_1 = K$ and

$$K_{i+1} = \{\xi\eta\,;\, \xi \in K_i, \eta = \wedge, 0 \quad\text{or}\quad 1\} \subset \Sigma$$

for $i = 1, 2, \cdots$. Then each K_i is a finite set. Since

$$\bigcup_{i=0}^{k} (W - i) \subset \{j\,;\, \beta(j+k) \in K_{k+1}\},$$

we have

$$\lim_{k\to\infty} \mu(\{(\gamma,\tau) \in F,\ \tau(1) \in K_{k+1}\}) = 1.$$

Thus μ is a probability measure on F and

$$\mu = \text{weak}\ \underset{\substack{n\in U\\ n\to\infty}}{\text{limit}}\ \frac{1}{n}\sum_{i=0}^{n-1} \delta_{(T^i\alpha, T^i\beta)}.$$

Clearly, $\mu|_{\{0,1\}^N} = \mu_\alpha^U = P^N$.

Let $\tilde F = \{(\gamma,\tau) \in \{0,1\}^Z \times \Sigma^Z;\ \tau(i+1) = \tau(i)\gamma(i)$ for any $i \in Z\}$. Let $\tilde\mu$ be the natural extension of μ on $\tilde F$; $\tilde\mu$ is the unique shift invariant measure on $\tilde F$ such that $\tilde\mu|_{\{0,1\}^N \times \Sigma^N} = \mu$. Let Q_1, Q_2 and Q_3 be the partitions on $\tilde F$ such that

$$Q_1 = \{\{(\gamma,\tau) \in \tilde{F};\ \gamma(0) = i\};\ i \in \{0,1\}\}$$

$$Q_2 = \{\{(\gamma,\tau) \in \tilde{F};\ \tau(0) = \xi\};\ \xi \in \Sigma\} \text{ and}$$

$$Q_3 = \{\{(\gamma,\tau) \in \tilde{F};\ \tau(0) = \xi\};\ \xi \in K\} \cup \{(\gamma,\tau) \in \tilde{F};\ \tau(0) \in K^c\}.$$

Then Q_1 and Q_3 are finite partitions. Since

$$\lim_{k\to\infty} \sigma_U\left(\bigcup_{i=0}^{k-1} (W-i)\right) = 1,$$

$$\tilde{\mu}(\{(\gamma,\tau);\ \gamma(i) \in K \text{ for infinitely many } i > 0\}) = 1.$$

This implies that

$$\bigvee_{i=1}^{\infty} T^i(Q_1 \vee Q_3) = \bigvee_{i=1}^{\infty} T^i(Q_1 \vee Q_2)$$

under $\tilde{\mu}$. Therefore,

$$Q_3 < Q_2 < \bigvee_{i=1}^{\infty} T^i(Q_1 \vee Q_3)$$

and

$$\bigvee_{i=1}^{\infty} T^i(Q_1 \vee Q_3) \vee Q_3 = \bigvee_{i=1}^{\infty} T^i(Q_1 \vee Q_2) \vee Q_2$$

under $\tilde{\mu}$. Hence,

$$\begin{aligned} H(Q_1) &= H\left(Q_1 \middle| \bigvee_{i=1}^{\infty} T^i Q_1\right) \\ &\leqq H\left(Q_1 \vee Q_3 \middle| \bigvee_{i=1}^{\infty} T^i(Q_1 \vee Q_3)\right) \\ &= H\left(Q_1 \middle| \bigvee_{i=1}^{\infty} T^i(Q_1 \vee Q_3) \vee Q_3\right) \\ &\quad + H\left(Q_3 \middle| \bigvee_{i=1}^{\infty} T^i(Q_1 \vee Q_3)\right) \\ &= H\left(Q_1 \middle| \bigvee_{i=1}^{\infty} T^i(Q_1 \vee Q_2) \vee Q_2\right) \end{aligned}$$

under $\tilde{\mu}$. Thus Q_1 is independent of $\bigvee_1^\infty T^i(Q_1 \vee Q_2) \vee Q_2$ under $\tilde{\mu}$.

Let $\Delta = \{(\gamma,\tau) \in F;\ \tau(1) \in K \text{ and } \tau(i) \in K \text{ for infinitely many } i \in N\}$. For $(\gamma,\tau) \in \Delta$, define $\psi(\gamma,\tau) \in \{0,1\}^N$ by $\psi(\gamma,\tau)\ (i) = \gamma(k_i)$, where $\{k_1 < k_2 < \cdots\} =$

$\{i\,;\tau(i)\in K\}$. For $\xi=(\xi_1,\xi_2,\cdots,\xi_n)\in\{0,1\}^n$, let $\Gamma_\xi=\{\gamma\in\{0,1\}^N;\ \gamma(i)=\xi_i$ for $i=1,2,\cdots,n\}$. For $\xi\in\{0,1\}^*$ and $(\gamma,\tau)\in F$, define

$$g_\xi(\gamma,\tau)=\begin{cases}1 & \text{if}\quad (\gamma,\tau)\in\Delta \quad\text{and}\quad \psi(\gamma,\tau)\in\Gamma_\xi\\ 0 & \qquad\qquad \text{else.}\end{cases}$$

Then it is easy to check that g_ξ is continuous at almost all points with respect to μ. Let V be a subset of $\{i\,;\,t_i\in U\}$ $(\subset V')$ such that $V\in\Xi_{\alpha_s}$. Let $\nu=\mu^{\,V}_{\alpha_s}$. We prove inductively that for any $\xi\in\{0,1\}^*$, $\nu(\Gamma_\xi)>0$ implies $\int g_\xi d\mu>0$ and $\nu(\Gamma_\xi)=P^N(\Gamma_\xi)$. In the case of $\xi=\wedge$, we have

$$\int g_\wedge d\mu=\lim_{\substack{n\in U\\ n\to\infty}}\frac{1}{n}\,|\{i\,;\,\beta(i)\in K,\ 1\leqq i\leqq n\}|=\sigma_U(W)>0 \ \text{ and}$$

$\nu(\Gamma_\wedge)=P^N(\Gamma_\wedge)=1$. Let $\xi\in\{0,1\}^*$ and $x\in\{0,1\}$ satisfy $\nu(\Gamma_{\xi x})>0$. Assume the above statement is true for ξ. Then since $\nu(\Gamma_\xi)>0$, we have $\int g_\xi d\mu>0$ and $\nu(\Gamma_\xi)=P^N(\Gamma_\xi)$. Hence,

$$\frac{\nu(\Gamma_{\xi x})}{\nu(\Gamma_\xi)}=\lim_{\substack{n\in V\\ n\to\infty}}\frac{\sum_{i=0}^{n-1}\chi_{\Gamma_{\xi x}}(T^i\alpha_s)}{\sum_{i=0}^{n-1}\chi_{\Gamma_\xi}(T^i\alpha_s)}$$

$$=\lim_{\substack{n\in U\\ n\to\infty}}\frac{\sum_{i=0}^{n-1}g_{\xi x}(T^i\alpha,T^i\beta)}{\sum_{i=0}^{n-1}g_\xi(T^i\alpha,T^i\beta)}$$

$$=\frac{\int g_{\xi x}d\mu}{\int g_\xi d\mu}\,.$$

Since $\nu(\Gamma_{\xi x})>0$, this implies $\int g_{\xi x}d\mu>0$. Moreover, since Q_1 is independent of $\bigvee_{i=1}^{\infty}T^i(Q_1\vee Q_2)\vee Q_2$ under $\tilde\mu$, $\int g_{\xi x}d\mu=\int g_\xi d\mu\cdot\mu(\{(\gamma,\tau);\ \gamma(1)=x\})$. Therefore, we have

$$\nu(\Gamma_{\xi x})=\nu(\Gamma_\xi)\cdot\mu(\{(\gamma,\tau);\ \gamma(1)=x\})$$

$$=\nu(\Gamma_\xi)\cdot P(x)=P^N(\Gamma_\xi)\cdot P(x)=P^N(\Gamma_{\xi x}).$$

Thus, we have proved that for any $\xi\in\{0,1\}^*$ such that $\nu(\Gamma_\xi)>0$, $\nu(\Gamma_\xi)=P^N(\Gamma_\xi)$. Since ν and P^N are both probability measures, this clearly implies that $\nu=P^N$. This completes the proof of Lemma 1. ■

As promised, we now show how to reduce the general case to the one considered in the lemma. Let $U' = \{t_i;\ i \in V'\}$. Take a subset U of U' such that $\sigma_U(\bigcup_{i=0}^{k}(W-i))$ exists for any $k = 0, 1, 2, \cdots$. For $j \in W \cap (W^c - 1)$, let

$$b(j) = \max\left\{k;\ j \in \bigcap_{i=1}^{k} (W^c - i)\right\}.$$

If $\{b(j)\}$ is bounded, then already any V' is such that U' satisfies the hypothesis of Lemma 1. We proceed then under the assumption that $b(j)$ is unbounded.

Take a sequence $j_1 < j_2 < \cdots$ of integers belonging to $W \cap (W^c - 1)$ such that

(i) $\lim_{n\to\infty} b(j_n) = \infty$ and

(ii) $\sigma_U\left(\bigcup_{i=1}^{\infty} \{j_i + 1, j_i + 2, \cdots, j_i + b(j_i)\}\right) = \lim_{k\to\infty} \sigma_U\left(\bigcap_{i=0}^{k} (W^c - i)\right).$

Let $L_1 = \{1, 2, \cdots, j_1\}$ and

$$L_i = \{j_{i-1} + b(j_{i-1}) + 1,\ j_{i-1} + b(j_{i-1}) + 2, \cdots, j_i\}$$

for $i = 2, 3, \cdots$. Denote

$$\alpha(\{n+1, n+2, \cdots, n+l\}) = (\alpha(n+1), \alpha(n+2), \cdots, \alpha(n+l)).$$

Define

$$\gamma = \alpha(L_1)B_1\alpha(L_2)B_2 \cdots \in \{0,1\}^N,$$

where for each $i \in N$, $B_i \in \{0,1\}^*$ is one of the shortest blocks such that

$$\alpha(1)\alpha(2)\cdots\alpha(j_i)B_i \sim_s \alpha(1)\alpha(2)\cdots\alpha(j_i + b(j_i)).$$

Note that the lengths of B_i's are bounded, since $\alpha(1)\alpha(2)\cdots\alpha(j_i - 1) \in S/\sim_s$, $\alpha(1)\alpha(2)\cdots\alpha(j_i + b(j_i)) \in S/\sim_s$ and $S/\sim_s$ is a finite set. Let D be the set of indices which are occupied by $\alpha(L_i)$'s in γ. Then it is clear that $\underline{\sigma}(D) = 1$, since $\underline{\sigma}(\bigcup_{i=1}^{\infty} L_i) \geqq \underline{\sigma}(W) > 0$, $\lim_{n\to 1} b(j_n) = \infty$ and the lengths of B_i's are bounded.

Generally, we say that β and β' belonging to $\{0,1\}^N$ are *equivalent* if there exists an

$$\begin{cases} E' = \{e'_1 < \cdots\} \subset N \\ E = \{e_1 < e_2 < \cdots\} \subset N \end{cases}$$

such that

$$\begin{cases} \underline{\sigma}(E') = 1 \\ \underline{\sigma}(E) = 1 \end{cases}$$

and $\beta(e_i) = \beta'(e'_i)$ for any $i \in N$. In this case, $\Xi_\beta = \Xi_{\beta'}$ and $\mu_\beta^R = \mu_{\beta'}^R$ hold for any $R \in \Xi_\beta$. In particular, β is normal if and only if β' is normal.

Note that γ and $\gamma' = \alpha(L_1)\alpha(L_2)\cdots$ are equivalent. Let $L = \bigcup_{i=1}^{\infty} L_i$. Then it holds that the support of $\mu_{\chi_L}^R$ is contained in the two point set $\{(0,0,\cdots), (1,1,\cdots)\}$ for any $R \in \Xi_{\chi_L}$ since $\underline{\sigma}(L) > 0$ and $\lim_{n\to\infty} b(j_n) = \infty$. Therefore χ_L is a completely deterministic sequence such that 1 appears with positive frequency. This implies that γ' is a normal number [2]. Hence γ is also a normal number. For $i \in L$, define $c(i) \in D$ by

$$|L \cap \{1,2,\cdots,i\}| = |D \cap \{1,2,\cdots,c(i)\}|.$$

Then it follows that $\alpha(i) = \gamma(c(i))$ and

$$\alpha(1)\alpha(2)\cdots\alpha(i-1) \sim_s \gamma(1)\gamma(2)\cdots\gamma(c(i)-1)$$

for any $i \in L$. This fact, combined with $W \subset L$, $\underline{\sigma}(W) > 0$ and $\underline{\sigma}(D) = 1$, implies that γ_s is a proper subsequence of γ and that α_s and γ_s are equivalent.

Let

$$W' = \{i;(\gamma(1),\gamma(2),\cdots,\gamma(i-1)) \in S\} = \{t'_1 < t'_0 < \cdots\}$$

$$V'' = \{i; t_i \in U\}$$

$$U'' = \{t'_i;\ i \in V''\} = \{u''_1 < u''_2 \cdots\} \quad \text{and}$$

$$U = \{u_1 < u_2 < \cdots\}.$$

Then we have $\lim_{i\to\infty}(u''_i/c(u_i)) = 1$ by the same reason mentioned above. Also, we can prove without difficulty using (ii) that

$$\lim_{k\to\infty} \sigma_{c(U)}\left(\bigcup_{i=0}^{k} (W'-i)\right) = 1,$$

and hence

$$\lim_{k\to\infty} \sigma_{U''}\left(\bigcup_{i=0}^{k} (W'-i)\right) = 1.$$

Thus we have found for any V' a subset $V'' \subset V'$ such that Lemma 1 applies, and this completes the proof of the theorem. ■

3. Examples and open questions

EXAMPLE 1. Let $S = \bigcup_{n=1}^{\infty}\{0,1\}^*\ 110^n 10^n$. Then $S/\sim_s$ consists of one element. Thus S preserves normality and clearly for any normal number α, α_s

is proper. Note that $\{0,1\}^*/\sim_s$ is an infinite set and hence the selection rule S cannot be duplicated by a finite automaton.

EXAMPLE 2. Let $S=\{(\xi_1,\xi_2,\cdots,\xi_n)\in\{0,1\}^*,\ \Sigma_{i=1}^n(-1)^{\xi_i}\geqq 0\}$. Then S preserves normality. This doesn't follow from our theorem, but rather from the following lemma which was suggested to us by Professor H. Furstenberg.

LEMMA 2. *Let μ be an ergodic measure for which*

$$\mu\left(\left\{\gamma\in\{0,1\}^N;\ \sigma\left(\left\{n;\ \sum_{i=1}^n(-1)^{\gamma(i)}\in K\right\}\right)=0\right\}\right)=1$$

for any finite set $K\subset Z$. Then in fact

$$(*)\qquad \sigma\left(\left\{n;\ \sum_{i=1}^n(-1)^{\alpha(i)}\in K\right\}\right)=0$$

for any finite set $K\subset Z$, and every α which is generic for μ.

PROOF. It is sufficient to prove (*) for the case that $K=\{0\}$. The general case then follows by considering $\xi\alpha$ instead of α where $\xi=0^{|k|}$, or $1^{|k|}$ according as $k\geqq 0$ or $k<0$, and $k\in K$.

Suppose then that (*) does not hold for some α which is generic for μ. Let $Z*=Z\cup\{\infty\}$ be the one point compactification of Z. Define $\beta\in Z*^N$ by $\beta(1)=0$ and $\beta(i)=\Sigma_{j=1}^{i-1}\alpha(j)$ for $i=2,3,\cdots$. Take any $U\subset N$ such that

$$\sigma_U\left(\left\{n;\ \sum_{i=1}^n(-1)^{\alpha(i)}=0\right\}\right)>0$$

and $U\in\Xi_{(\alpha,\beta)}$. Let $\nu=\mu^U_{(\alpha,\beta)}$. Then we have

$$\nu\mid_{\{0,1\}^N}=\mu,\ \nu(\{(\gamma,\tau);\ \tau(0)=0\})>0$$

and the support of ν is contained in a closed shift invariant set

$$F=\{(\gamma,\tau)\in\{0,1\}^N\times Z*^N;\ \tau(i+1)=(-1)^{\gamma(i)}+\tau(i)\text{ for any }i\in N\}.$$

This implies that there exists an *ergodic* measure which satisfies the three conditions just mentioned. We will denote this ergodic measure by the same symbol ν. From the ergodicity and $\nu(\{(\gamma,\tau);\ \tau(0)=0\})>0$, we have that $\sigma(\{n;\ \tau(n+1)=0\})>0$ for almost all (γ,τ) with respect to ν. Let ν_0 be the measure on $\{0,1\}^N$ defined by

$$\nu_0(A)=\frac{\nu(A\times\{\tau\in Z*^N;\ \tau(0)=0\})}{\nu(\{0,1\}^N\times\{\tau\in Z*^N;\ \tau(0)=0\})}$$

for any Borel set $A \subset \{0,1\}^N$. Then

$$\sigma\left(\left\{n ; \sum_{i=1}^{n} (-1)^{\gamma(i)} = 0\right\}\right) > 0$$

holds for almost all $\gamma \in \{0,1\}^N$ with respect to ν_0. Since ν_0 is absolutely continuous with respect to μ, this statement is also true with respect to μ, which contradicts our assumption.

Proof of Example 2. Let α be a normal number such that α_s is a proper subsequence. Let $W = \{i ; (\alpha(1), \alpha(2), \cdots, \alpha(i-1)) \in S\}$. Since

$$\sigma\left(\left\{n ; \sum_{i=1}^{n} (-1)^{\alpha(i)} = 0\right\}\right) = 0$$

by Lemma 2, $\chi_W \in \{0,1\}^N$ is completely deterministic. Since 1 appears with positive density in χ_W, α_s is a normal number [2].

Example 3. The theorem contains the case of selection rule S which satisfies $SS \subset S$ and $S^{-1}S (= \{\eta ; \xi\eta \in S$ holds for some $\xi \in S\}) \subset S$. If S fails to satisfy $S^{-1}S \subset S$, then S does not necessarily preserve normality even though $SS \subset S$. Let $\alpha \in \{0,1\}^N$ be a normal number such that $\alpha(1) = 0$. Let

$$S = \left\{(\xi_1, \xi_2, \cdots, \xi_n) \in \{0,1\}^* ; \sum_{i=1}^{n} \frac{\xi_i}{2^i} + \frac{1}{2^{n+1}} \leqq \sum_{i=1}^{\infty} \frac{\alpha(i)}{2^i}\right\}.$$

Then $SS \subset S$ holds. But S does not preserve normality, since $\alpha_s = (1,1,1,\cdots)$ is a proper subsequence of α and is not normal.

Open Questions

1. Does $S \subset \{0,1\}^*$ preserve normality if it is a context-free language?

2. Does S preserve normality if the increasing order of $|S/\sim_s \cap \{0,1\}^n|$ as n tends to infinity is small enough, say $\log n$?

3. Given integers $z_0 = 1$ and z_1. Find conditions on $U \subset Z$ under which the selection rule $S = \{(\xi_1, \xi_2, \cdots, \xi_n) \in \{0,1\}^*, \ \Sigma_{i=1}^n z_{\xi_i} \in U\}$ preserves normality. We know a necessary and sufficient condition in case that $z_1 = 1$. On the other hand, if U is an arithmetic sequence then S is generated by a finite automaton. The general case seems to be wide open.

REFERENCES

1. Y. N. Agafonov, *Normal sequences and finite automata*, Dokl. Akad. Nauk SSSR **179** (1968).

2. Teturo Kamae, *Subsequences of normal sequences*, Israel J. Math. **16** (1973), 121–149.

3. B. Weiss, *Normal numbers as collectives*, Proc. of Symp. on Ergodic Theory, Univ. of Kentucky, 1971.

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